

FREE DIFFUSIONS AND PROPERTY AO

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1. INTRODUCTION

Guionnet and Shlyakhtenko extended Langevin-type free stochastic dynamics to the case of operators interacting by a locally convex potential in [GS], and among other things used these results to give technical properties of certain operator algebras. They were specifically interested in algebras generated by the stationary laws of free stochastic differential equations (SDE) of the form

$$dX_t = dS_t - \frac{1}{2}DV(X_t)dt$$

for a suitably locally convex multivariable *-polynomial V .

Indeed, they were able to establish that such an SDE has a unique stationary distribution μ_V satisfying the Schwinger-Dyson equation

$$\mu_V \otimes \mu_V(\partial_i P) = \mu_V(D_i V P)$$

where ∂_i is the non-commutative partial difference quotient and D_i is the cyclic partial derivative. By using the fact that they also had convergence in norm to this distribution from all initial data, they were able to show that the von Neumann algebra M_V generated by operators with joint law μ_V is a factor with the Haagerup property. They also proved that M_V has finite free entropy dimension and hence is prime and has no Cartan subalgebras. All of this provided evidence for the conjecture of Voiculescu that M_V is isomorphic to a free group factor.

Recall that a von Neumann algebra $M \subseteq \mathbb{B}(H)$ is said to have property AO if there are ultraweakly dense C^* subalgebras $A \subseteq M$ and $B \subseteq M'$ with A locally reflexive and such that the *-homomorphism $\Phi : A \otimes B \rightarrow \mathbb{B}(H)/\mathbb{K}$ given by

$$\Phi\left(\sum a_i \otimes b_i\right) = \pi\left(\sum a_i b_i\right)$$

is continuous with respect to the minimal tensor norm.

In this short paper we will demonstrate how the above techniques can be used to prove that M_V has this property AO of Ozawa, and is thus solid by Theorem 6 in [Oz]. This result adds to the evidence of the conjecture above as solidity (and AO) are well-known properties of free group factors.

2. PRELIMINARIES

Denote the set of polynomials in m non-commuting indeterminates (X_1, \dots, X_m) by $\mathbb{C}\langle X_1, \dots, X_m \rangle$. Consider this as a subalgebra of $\mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$, the *-algebra of polynomials in the m indeterminates and their formal adjoints. We will say that a polynomial $V \in \mathbb{C}\langle X_1, \dots, X_m \rangle$ is self-adjoint if $V(X_1, \dots, X_m)^* = V(X_1^*, \dots, X_m^*)$.

Recall the cyclic gradient D of Rota, Sagan, and Stein, which is linear and given on any noncommutative multinomial $P \in \mathbb{C}\langle X_1, \dots, X_m \rangle$ by $DP = (D_1P, \dots, D_mP)$ with

$$D_iP = \sum_{P=QX_iR} RQ.$$

Next, recall the non-commutative difference quotient $\partial : \mathbb{C}\langle X_1, \dots, X_m \rangle \rightarrow \mathbb{C}\langle X_1, \dots, X_m \rangle \otimes \mathbb{C}\langle X_1, \dots, X_m \rangle$ which is again linear and given on a multinomial P by $\partial P = (\partial_1P, \dots, \partial_mP)$ with

$$\partial_iP = \sum_{P=QX_iR} Q \otimes R.$$

Finally, define for a pair of m -tuples of elements $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_m)$ in any $*$ -algebra the notation

$$X.Y = \frac{1}{2} \sum_{i=1}^m (X_i Y_i^* + Y_i X_i^*).$$

Then, for $V \in \mathbb{C}\langle X_1, \dots, X_m \rangle$, we say that V is (c, M) -convex if for any m -tuples $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_m)$ of operators in any C^* algebra A satisfying $\|X\|_A, \|Y\|_A \leq M$ we have

$$(DV(X) - DV(Y)).(X - Y) \geq c(X - Y).(X - Y).$$

(This is to be understood as an operator inequality in A .)

We will be considering the solutions to the multivariable SDE

$$dX_t = dS_t - \frac{1}{2}DV(X_t)dt$$

for V a (c, M) -convex polynomial. As such, we will fix an ambient free probability space (\mathcal{A}, τ) generated by a free Brownian motion S . For more information on SDE and free Brownian motion, see [BS, BS2].

Let V be a (c, M) -convex polynomial in $\mathbb{C}\langle X_1, \dots, X_m \rangle$ and recall the following result ([GS] Lemma 2.1):

Theorem 2.1. *There exist finite constants*

$$M_0 = M_0(c, \|DV(0).DV(0)\|),$$

$$B_0 = B_0(c, \|DV(0).DV(0)\|),$$

and

$$b = b(c, \|DV(0).DV(0)\|, M) \geq B_0$$

so that whenever $M \geq M_0$ and Z is an m -tuple with $\|Z\| < b$, there exists a unique solution X_t^Z to the SDE

$$dX_t^Z = dS_t - \frac{1}{2}DV(X_t^Z)dt, \quad t \in [0, +\infty)$$

with the initial data $X_0^Z = Z$.

Moreover, in this case,

$$\begin{aligned} \|X_t^Z\| &\leq M, & \forall t \in [0, +\infty), \\ \limsup_{t \rightarrow \infty} \|X_t^Z\| &\leq B_0, \\ X_t^Z &\in C^*(Z, S_q : q \in [0, t]), & \forall t \in [0, +\infty). \end{aligned}$$

If V is self-adjoint (c, M) -convex and $X_0 = Z$ is self-adjoint, then the above results hold and additionally, X_t remains self-adjoint for all $t \geq 0$.

Thus, if V is sufficiently locally convex, and our initial data is appropriately bounded, we then have a unique bounded solution that exists for all time to our desired free diffusion equation. Now, let V be as above and assume additionally that V is self-adjoint. If we change the initial data for our SDE, we have the following asymptotic uniqueness result ([GS] Theorem 2.2):

Theorem 2.2. *Let M_0, B_0 and b be as in Theorem 2.1, and assume that $M \geq M_0$, and that Z is an m -tuple of operators with $\|Z\|_\infty < b$. Consider the unique solutions X_t^Z, X_t^0 to the free SDE*

$$dX_t = dS_t - \frac{1}{2}DV(X_t)dt$$

with initial conditions $X_0^Z = Z$, and $X_0^0 = 0$ respectively. Then

- (i) $\|X_t^Z - X_t^0\|_\infty \rightarrow 0$ as $t \rightarrow \infty$.
- (ii) *The law of (X_t^Z) converges to a stationary law $\mu_V \in \mathbb{C}\langle X_1, \dots, X_m \rangle'$ which satisfies for all $P \in \mathbb{C}\langle X_1, \dots, X_m \rangle$*

$$(1) \quad \sum_{i=1}^m \mu_V \otimes \mu_V (\partial_i D_i P) = \sum_{i=1}^m \mu_V (D_i V D_i P).$$

Moreover, for all $i \in \{1, \dots, m\}$,

$$\mu_V(X_i^k) \leq B_0^k.$$

Any law $\nu \in \mathbb{C}\langle X_1, \dots, X_m \rangle'$ of variables bounded in operator norm by b that satisfies (1) is such that $\nu = \mu_V$.

3. RESULT

Assume that V is a self-adjoint (c, M) -convex polynomial in m variables with $M \geq M_0$ for M_0 as above.

Theorem 3.1. *Let Z be an m -tuple of self-adjoint operators satisfying $\|Z\|_\infty < b$ and having the stationary law μ_V from above. Then $W^*(Z)$ has Ozawa's property AO, and is hence solid.*

We first establish some notation and prove two lemmas.

Let $N = W^*(Z)$, $A = C^*(Z)$, and $M = W^*(Z, S_t : t \geq 0)$. Endow M with the trace τ from the ambient free probability space, and let $H = L^2(M, \tau)$. Denote the unique solutions to the free SDE

$$dX_t = dS_t - \frac{1}{2}DV(X_t)dt$$

with initial data 0 and Z by X_t^0 and X_t^Z , respectively. As the law of X_t^Z is stationary, we have, for each t , homomorphic embeddings $\theta_t : N \rightarrow M$ satisfying $\theta_t(Z) = X_t^Z$. Let $B = J_H A J_H$, and define $\tilde{\theta}_t(J_H a J_H) = J_H \theta_t(a) J_H$ for $a \in A$.

Lemma 3.2. *Let $\sum_{i=1}^n a_i \otimes b_i$ be an element in the algebraic tensor product $A \otimes B$. For every $\epsilon > 0$ there exists $p_i, q_i \in \mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$ and $t_0 \geq 0$ such that for all $t \geq t_0$,*

$$\left\| \sum_{i=1}^n \theta_t(a_i) \otimes \tilde{\theta}_t(b_i) - \sum_{i=1}^n p_i(X_t^0) \otimes J_H q_i(X_t^0) J_H \right\|_{\min} < \epsilon$$

and

$$\left\| \sum_{i=1}^n \theta_t(a_i) \tilde{\theta}_t(b_i) - \sum_{i=1}^n p_i(X_t^0) J_H q_i(X_t^0) J_H \right\|_{\infty} < \epsilon$$

where $\|\cdot\|_{\min}$ denotes the minimal tensor norm on $M \otimes J_H M J_H$.

Proof: First note that if $\delta > 0$ and $a \in A$, then we can find a $p \in \mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$ and a $t_0 \geq 0$ such that for all $t \geq t_0$,

$$\|\theta_t(a) - p(X_t^0)\|_{\infty} < \delta.$$

Indeed, first choose p so that

$$\|a - p(Z)\|_{\infty} < \delta/2.$$

Then, as by Theorem 2.2 we have that $\|X_t^Z - X_t^0\|_{\infty} \rightarrow 0$ as $t \rightarrow \infty$, choose t_0 so that for all $t \geq t_0$,

$$\|\theta_t(p(Z)) - p(X_t^0)\|_{\infty} = \|p(X_t^Z) - p(X_t^0)\|_{\infty} < \delta/2.$$

The triangle inequality then implies that these are the desired p and t_0 .

Note also that conjugation by J_H shows that we can obtain a similar corresponding statement for approximating $\tilde{\theta}_t(b)$ by $J_H q(X_t^0) J_H$ for $q \in \mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$ and t sufficiently large.

Now, fix $1 \geq \epsilon > 0$, and for each i choose $p_i, q_i \in \mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$ and t such that for all $t \geq t_0$,

$$\|\theta_t(a_i) - p_i(X_t^0)\|_{\infty} < \frac{\epsilon}{2n(\|b_i\|_{\infty} + 1)}$$

and

$$\|\tilde{\theta}_t(b_i) - J_H q_i(X_t^0) J_H\|_{\infty} < \frac{\epsilon}{2n(\|a_i\|_{\infty} + 1)}.$$

Then we have by a simple estimate that

$$\|\theta_t(a_i) \otimes \tilde{\theta}_t(b_i) - p_i(X_t^0) \otimes J_H q_i(X_t^0) J_H\|_{\min} < \epsilon/n,$$

and

$$\|\theta_t(a_i) \tilde{\theta}_t(b_i) - p_i(X_t^0) J_H q_i(X_t^0) J_H\|_{\infty} < \epsilon/n.$$

Thus the triangle inequality again implies the lemma. \square

Lemma 3.3. *There exists $\alpha > 0$ such that for any $p_i, q_i \in \mathbb{C}\langle X_1, \dots, X_m, X_1^*, \dots, X_m^* \rangle$ where $i \in \{1, \dots, n\}$, any $\epsilon > 0$, and any $t \geq 0$ there exists a compact operator T such that*

$$\left\| \sum_{i=1}^n p_i(X_t^0) J_H q_i(X_t^0) J_H + T \right\|_{\infty} \leq \alpha \left\| \sum_{i=1}^n p_i(X_t^0) \otimes J_H q_i(X_t^0) J_H \right\|_{\min} + \epsilon.$$

Proof: Let $C = C^*(S_t : t \geq 0)$, $D = J_H C^*(S_t : t \geq 0) J_H$, and note that C is nuclear. Thus the *-homomorphism $\Phi : C \otimes D \rightarrow \mathbb{B}(H)/\mathbb{K}$ given by

$$\Phi \left(\sum c_i \otimes d_i \right) = \pi \left(\sum c_i d_i \right)$$

(for π the canonical homomorphism into the Calkin algebra) is continuous with respect to the minimal tensor norm on $C \otimes D$.

So, there exists an $\alpha > 0$ such that for every $c_1, \dots, c_n \in C$, $d_1, \dots, d_n \in D$ and $\epsilon > 0$ there exists a compact T_ϵ such that

$$\left\| \sum_{i=1}^n c_i d_i + T_\epsilon \right\|_\infty \leq \alpha \left\| \sum_{i=1}^n c_i \otimes d_i \right\|_{\min} + \epsilon$$

Note that by Theorem 2.1 we have that $X_t^0 \in C$ for any t , and so letting $c_i = p_i(X_t^0)$, $d_i = J_H q_i(X_t^0) J_H$ and setting $T = T_\epsilon$ proves the lemma. \square

Proof of Theorem 3.1: We will show that the *-homomorphism $\Psi : A \otimes B \rightarrow \mathbb{B}(L^2(N, \tau))/\mathbb{K}$ given by

$$\Psi \left(\sum a_i \otimes b_i \right) = \pi \left(\sum a_i b_i \right)$$

is continuous with respect to the minimal tensor norm on $A \otimes B$. (Here we have identified B with its restriction to $L^2(N, \tau) \subset H$.) Note that this indeed suffices as by Lemma 4.3 in [GS], A is exact and hence locally reflexive.

To this end, let $\epsilon > 0$ and let α be as in Lemma 3.3. By Lemma 3.2 choose p_i , q_i and t such that

$$\left\| \sum_{i=1}^n \theta_t(a_i) \otimes \tilde{\theta}_t(b_i) - \sum_{i=1}^n p_i(X_t^0) \otimes J_H q_i(X_t^0) J_H \right\|_{\min} < \frac{\epsilon}{2(\alpha + 1)}$$

and

$$\left\| \sum_{i=1}^n \theta_t(a_i) \tilde{\theta}_t(b_i) - \sum_{i=1}^n p_i(X_t^0) J_H q_i(X_t^0) J_H \right\|_\infty < \frac{\epsilon}{2(\alpha + 1)}.$$

Then, apply Lemma 3.3 to find a compact operator T such that

$$\left\| \sum_{i=1}^n p_i(X_t^0) J_H q_i(X_t^0) J_H + T \right\|_\infty \leq \alpha \left\| \sum_{i=1}^n p_i(X_t^0) \otimes J_H q_i(X_t^0) J_H \right\|_{\min} + \epsilon/2.$$

We then have by another simple estimate that

$$\left\| \sum_{i=1}^n \theta_t(a_i) \tilde{\theta}_t(b_i) + T \right\|_\infty \leq \alpha \left\| \sum_{i=1}^n \theta_t(a_i) \otimes \tilde{\theta}_t(b_i) \right\|_{\min} + \epsilon.$$

If we restrict the operator on the left-hand side to $L^2(\theta_t(N), \tau)$, we will obtain the inequality

$$\left\| \sum_{i=1}^n \theta_t(a_i) \tilde{\theta}_t(b_i) + S \right\|_\infty \leq \alpha \left\| \sum_{i=1}^n \theta_t(a_i) \otimes \tilde{\theta}_t(b_i) \right\|_{\min} + \epsilon$$

for $S = e_N T e_N$ a compact operator on $L^2(\theta_t(N), \tau)$. By identifying N with $\theta_t(N)$ and $L^2(N, \tau)$ with $L^2(\theta_t(N), \tau)$, we thus get a compact operator R on $L^2(N, \tau)$

$$\left\| \sum_{i=1}^n a_i b_i + R \right\|_{\infty} \leq \alpha \left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} + \epsilon$$

and this proves the theorem. \square

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